CHARACTERIZATIONS OF RIGHT MODULAR GROUPOIDS BY $(\in, \in \vee q_k)$-FUZZY IDEALS

Madad Khan and Shamas-ur-Rehman
Department of Mathematics
COMSATS Institute of Information Technology
Abbottabad, Pakistan.
E-mail: madadmath@yahoo.com
E-mail: shamas_200814@yahoo.com

Abstract. In this paper, we have introduced the concept of $(\in, \in \vee q)$-fuzzy ideals in a right modular groupoid. We have discussed several important features of a completely regular right modular groupoid by using the $(\in, \in \vee q)$-fuzzy left (right, two-sided) ideals, $(\in, \in \vee q)$-fuzzy (generalized) bi-ideals and $(\in, \in \vee q)$-fuzzy $(1,2)$-ideals. We have also used the concept of $(\in, \in \vee q_k)$-fuzzy left (right, two-sided) ideals, $(\in, \in \vee q_k)$-fuzzy quasi-ideals $(\in, \in \vee q_k)$-fuzzy bi-ideals and $(\in, \in \vee q_k)$-fuzzy interior ideals in completely regular right modular groupoid and proved that the $(\in, \in \vee q_k)$-fuzzy left (right, two-sided), $(\in, \in \vee q_k)$-fuzzy (generalized) bi-ideals, and $(\in, \in \vee q_k)$-fuzzy interior ideals coincide in a completely regular right modular groupoid.
Keywords. Right modular groupoid, completely regular, $(\in, \in \vee q)$-fuzzy ideals and $(\in, \in \vee q_k)$-fuzzy ideals

Introduction

The fundamental concept of fuzzy sets was first introduced by Zadeh [18] in 1965. Given a set X, a fuzzy subset of X is, by definition an arbitrary mapping $f : X \to [0,1]$ where $[0,1]$ is the unit interval. Rosenfeld introduced the definition of a fuzzy subgroup of a group [15]. Kuroki initiated the theory of fuzzy bi ideals in semigroups [8]. The thought of belongingness of a fuzzy point to a fuzzy subset under a natural equivalence on a fuzzy subset was defined by Murali [11]. The concept of quasi-coincidence of a fuzzy point to a fuzzy set was introduce in [14]. Jun and Song introduced (α, β)-fuzzy interior ideals in semigroups [4].

In this paper we have characterized non-associative algebraic structures called right modular groupoids by their $(\in, \in \vee q_k)$-fuzzy ideals. A right modular groupoid M is non-associative and non-commutative algebraic structure mid way between a groupoid and a commutative semigroup.

The concept of a left almost semigroup (LA-semigroup) [5] or a right modular groupoid was first given by M. A. Kazim and M. Naseeruddin in 1972. A right modular groupoid M is a groupoid having the left invertive law,

$$(ab)c = (cb)a, \text{ for all } a,\, b,\, c \in M. \tag{1}$$

In a right modular groupoid M, the following medial law [5] holds,

$$(ab)(cd) = (ac)(bd), \text{ for all } a,\, b,\, c,\, d \in M. \tag{2}$$

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The left identity in a right modular groupoid if exists is unique [12]. In a right modular groupoid M with left identity the following paramedial law holds [13],

$$(ab)(cd) = (dc)(ba), \text{ for all } a, b, c, d \in M. \tag{3}$$

If a right modular groupoid M contains a left identity, then,

$$a(bc) = b(ac), \text{ for all } a,\ b,\ c \in M. \tag{4}$$

Preliminaries

Let M be a right modular groupoid, by a subgroupoid of M, we means a non-empty subset A of M such that $A^2 \subseteq A$. A non-empty subset A of a right modular groupoid M is called left (right) ideal of M if $MA \subseteq A$ $(AM \subseteq A)$. A is called two-sided ideal or simply ideal if it is both a left and a right ideal of M. A non empty subset A of a right modular groupoid M is called generalized bi-ideal of M if $(AM)A \subseteq A$. A subgroupoid A of M is called bi-ideal of M if $(AM)A \subseteq A$. A subgroupoid A of M is called interior ideal of M if $(MA)M \subseteq A$. A non-empty subset A of a right modular groupoid M is called quasi-ideal of M if $QM \cap MQ \subseteq Q$. Every one sided ideal is quasi ideal, every quasi ideal is, every bi-ideal is generalized bi-ideal but converse is not true in general. Also every two sided ideal is interior ideal but converse is not true.

Definition 1. *A fuzzy subset F of a right modular groupoid M is called a fuzzy interior ideal of M if it satisfy the following conditions,*
(i) $F(xy) \geq \min\{F(x), F(y)\}$ *for all* $x, y \in M$.
(ii) $F((xa)y) \geq F(a)$ *for all* $x, a, y \in M$.

Definition 2. *For a fuzzy set F of a right modular groupoid M and $t \in (0, 1]$, the crisp set $U(F; t) = \{x \in M$ such that $F(x) \geq t\}$ is called level subset of F.*

Definition 3. *A fuzzy subset F of a right modular groupoid M of the form*

$$F(y) = \begin{cases} t \in (0,1] & \text{if } y = x \\ 0 & \text{if } y \neq x \end{cases}$$

is said to be a fuzzy point with support x and value t and is denoted by x_t.

A fuzzy point x_t is said to *belong* to (*resp. quasi-coincident with*) a fuzzy set F, written as $x_t \in F$ (*resp.* $x_t qF$) if $F(x) \geq t$ (*resp.* $F(x) + t > 1$). If $x_t \in F$ or (*resp. and*) $x_t qF$, then we write $x_t \in \vee q$ $(\in \wedge q)F$. The symbol $\overline{\in \vee q}$ means $\in \vee q$ does not hold.

Lemma 1. *(cf. [7]) A fuzzy set F of a right modular groupoid M is a fuzzy interior ideal of M if and only if $U(F; t)\, (\neq \emptyset)$ is an interior ideal of M.*

Definition 4. *A fuzzy set F of a right modular groupoid M is called an $(\in, \in \vee q)$-fuzzy interior ideal of M if for all $t, r \in (0, 1]$ and $x, a, y \in M$.*
(A1) $x_t \in F$ *and* $y_r \in F$ *implies that* $(xy)_{\min\{t,r\}} \in \vee qF$.
(A2) $a_t \in F$ *implies* $((xa)y)_t \in \vee qF$

Definition 5. *A fuzzy set F of a right modular groupoid M is called an $(\in, \in \vee q)$-fuzzy bi-ideal of M if for all $t, r \in (0, 1]$ and $x, y, z \in M$.*
(B1) $x_t \in F$ *and* $y_r \in F$ *implies that* $(xy)_{\min\{t,r\}} \in \vee qF$.
(B2) $x_t \in F$ *and* $z_r \in F$ *implies* $((xy)z)_{\min\{t,r\}} \in \vee qF$.

Lemma 2. *A fuzzy set F of a right modular groupoid M is an $(\in, \in \vee q)$-fuzzy interior ideal of M if and only if $U(F;t)\,(\neq \emptyset)$ is an interior ideal of M, for all $t \in (0, 0 \cdot 5]$.*

Proof. Let F be an $(\in, \in \vee q)$-fuzzy interior ideal of M. Let $x, y \in U(F;t)$ and $t \in (0, 0 \cdot 5]$, then $F(x) \geq t$ and $F(y) \geq t$, so $F(x) \wedge F(y) \geq t$. As F is an $(\in, \in \vee q)$-fuzzy interior ideal of M, so

$$F(xy) \geq F(x) \wedge F(y) \wedge 0 \cdot 5 \geq t \wedge 0 \cdot 5 = t.$$

Therefore, $xy \in U(F;t)$. Now if $x, y \in M$ and $a \in U(F;t)$ then $F(a) \geq t$ then $F((xa)y) \geq F(a) \geq t$. Therefore $((xa)y) \in U(F;t)$ and $U(F;t)$ is an interior ideal.

Conversely assume that $U(F;t)$ is a fuzzy interior ideal of M. If $x, y \in U(F;t)$ then $F(x) \geq t$ and $F(y) \geq r$ which shows $x_t \in F$ and $y_r \in F$ as $U(F;t)$ is an interior ideal so $xy \in U(F;t)$ therefore $F(xy) \geq \min\{t, r\}$ implies that $(xy)_{\min\{t,r\}} \in F$, so $(xy)_{\min\{t,r\}} \in \vee qF$. Again let $x, y \in M$ and $a \in U(F;t)$ then $F(a) \geq t$ implies that $a_t \in F$ and $U(F;t)$ is an interior ideal so $((xa)y) \in U(F;t)$ then $F((xa)y) \geq t$ implies that $((xa)y)_t \in F$ so $((xa)y)_t \in \vee qF$. Therefore F is an $(\in, \in \vee q)$-fuzzy interior ideal. □

Theorem 1. *(cf. [7]) For a fuzzy set F of a right modular groupoid M. The conditions $(A1)$ and $(A2)$ of Definition 4, are equivalent to the following,*

$(A3)$ $(\forall x, y \in M) F(xy) \geq \min\{F(x), F(y), 0 \cdot 5\}$

$(A4)$ $F((xa)y) \geq \min\{F(a), 0 \cdot 5\}$.

Theorem 2. *For a fuzzy set F of a right modular groupoid M. The conditions $(B1)$ and $(B2)$ of Definition 5, are equivalent to the following,*

$(B3)$ $(\forall x, y \in M) F(xy) \geq \min\{F(x), F(y), 0 \cdot 5\}$

$(B4)$ $(\forall x, y, z \in M) F((xy)z) \geq \min\{F(x), F(y), 0 \cdot 5\}$.

Proof. It is similar to proof of theorem 1. □

Definition 6. *A fuzzy subset F of a right modular groupoid M is called an $(\in, \in \vee q)$-fuzzy $(1,2)$ ideal of M if*

(i) $F(xy) \geq \min\{F(x), F(y), 0.5\}$,

(ii) $F((xa)(yz)) \geq \min\{F(x), F(y), F(z), 0.5\}$, *for all $x, a, y, z \in M$.*

Theorem 3. *Every $(\in, \in \vee q)$-fuzzy bi-ideal is an $(\in, \in \vee q)$-fuzzy $(1,2)$ ideal of a right modular groupoid M, with left identity.*

Proof. Let F be an $(\in, \in \vee q)$-fuzzy bi-ideal of M and let $x, a, y, z \in M$ then by using (4) and (1), we have

$$\begin{aligned} F((xa)(yz)) &= F(y((xa)z)) \geq \min\{F(y), F((xa)z), 0.5\} \\ &= \min\{F(y), F((za)x), 0.5\} \geq \min\{(Fy), F(z), F(x), 0.5, 0.5\} \\ &= \min\{(Fy), F(z), F(x), 0.5\}. \end{aligned}$$

Therefore F is an $(\in, \in \vee q)$-fuzzy $(1,2)$ ideal of a right modular groupoid M. □

Theorem 4. *Every $(\in, \in \vee q)$-fuzzy interior ideal is an $(\in, \in \vee q)$-fuzzy $(1,2)$ ideal of a right modular groupoid M, with left identity e.*

Proof. Let F be an $(\in, \in \vee q)$-fuzzy interior ideal of M and let $x, a, y, z \in M$ then by using (1), we have

$$\begin{aligned} F((xa)(yz)) &\geq \min\{F(xa), F(yz), 0.5\} \geq \min\{F(xa), F(y), F(z), 0.5, 0.5\} \\ &= \min\{F((ex)a), F(y), F(z), 0.5\} = \min\{F((ax)e), F(y), F(z), 0.5\} \\ &\geq \min\{F(x), F(y), F(z), 0.5, 0.5\} = \min\{F(x), F(y), F(z), 0.5\}. \end{aligned}$$

Therefore F is an $(\in, \in \vee q)$ $-$fuzzy $(1,2)$ ideal of a right modular groupoid M. □

Theorem 5. *Let $\Phi : M \longrightarrow M'$ be a homomorphism of right modular groupoids and F and G be $(\in, \in \vee q)$-fuzzy interior ideals of M and M',respectively. Then*
(i) $\Phi^{-1}(G)$ is an $(\in, \in \vee q)$-fuzzy interior ideal of M.
(ii) If for any subset X of M there exist $x_\circ \in X$ such that $F(x_\circ) = \bigvee \{F(x) \mid x \in X\}$, then $\Phi(F)$ is an $(\in, \in \vee q)$-fuzzy interior ideal of M' when Φ is onto.

Proof. It is same as in [4]. □

Completely Regular Right Modular Groupoids

Definition 7. *A right modular groupoid M is called regular, if for each $a \in M$ there exist $x \in M$ such that $a = (ax)a$.*

Definition 8. *A right modular groupoid M is called left (right) regular, if for each $a \in M$ there exist $z \in M$ $(y \in M)$ such that $a = za^2$ $\left(a = a^2 y\right)$.*

Definition 9. *A right modular groupoid M is called completely regular if it is regular, left regular and right regular.*

Example 1. *Let $M = \{1, 2, 3, 4\}$ and the binary operation "$\circ$" defined on M as follows:*

$\circ$	1	2	3	4
1	4	1	2	3
2	3	4	1	2
3	2	3	4	1
4	1	2	3	4

Then clearly $(M, \circ)$ is a completely regular right modular groupoid with left identity 4.

Theorem 6. *If M is a right modular groupoid with left identity, then it is completely regular if and only if $a \in (a^2M)a^2$.*

Proof. Let M be a completely regular right modular groupoid with left identity, then for each $a \in M$ there exist $x, y, z \in M$ such that $a = (ax)a$, $a = a^2 y$ and $a = za^2$, so by using (1), (4) and (3), we get

$$\begin{aligned} a &= (ax)a = ((a^2y)x)(za^2) = ((xy)a^2)(za^2) = ((za^2)a^2)(xy) \\ &= ((a^2a^2)z)(xy) = ((xy)z)(a^2a^2) = a^2(((xy)z)a^2) \\ &= (ea^2)(((xy)z)a^2) = (a^2((xy)z))(a^2e) = (a^2((xy)z))((aa)e) \\ &= (a^2((xy)z))((ea)a) = (a^2((xy)z))a^2 \in (a^2M)a^2. \end{aligned}$$

Conversely, assume that $a \in (a^2M)a^2$ then clearly $a = a^2y$ and $a = za^2$, now using $(3), (1)$ and (4), we get

$$\begin{aligned} a &\in (a^2M)a^2 = \left(a^2M\right)(aa) = (aa)\left(Ma^2\right) = (aa)\left(M\left(aa\right)\right) \\ &= (aa)\left((eM))(aa)\right) = (aa)\left((aa)\left(Me\right)\right) \subseteq (aa)\left((aa)\,M\right) \\ &= (aa)\left(a^2M\right) = \left(\left(a^2M\right)a\right)a = \left(\left((aa)\,M\right)a\right)a = \left((aM)\,(aa)\right)a \\ &= \left(a\left((Ma)\,a\right)\right)a \subseteq (aM)a. \end{aligned}$$

Therefore M is completely regular. □

Theorem 7. *If M is a completely regular right modular groupoid, then every $(\in, \in \vee q)$-fuzzy $(1,2)$ ideal of M is an $(\in, \in \vee q)$-fuzzy bi-ideal of M.*

Proof. Let M be a completely regular and F is an $(\in, \in \vee q)$-fuzzy $(1,2)$ ideal of M. Then for $x \in M$ there exist $b \in M$ such that $x = (x^2b)x^2$, so by using (1) and (4), we have

$$\begin{aligned} F((xa)y) &= F((((x^2b)x^2)a)y) = F((ya)((x^2b)x^2)) \\ &\geq \min\left\{F(y), F(x^2b), F(x^2), 0.5\right\} \\ &\geq \min\left\{F(y), F(x^2b), F(x), F(x), 0.5, 0.5\right\} \\ &= \min\left\{F(y), F(x^2b), F(x), 0.5\right\} \\ &= \min\left\{F((xx)b), F(x), F(y), 0.5\right\} \\ &= \min\left\{F((bx)x), F(x), F(y), 0.5\right\} \\ &\geq \min\left\{F(bx), F(x), 0.5, F(x), F(y), 0.5\right\} \\ &= \min\left\{F(bx).F(x), F(y), 0.5\right\} \\ &= \min\left\{F(b((x^2b)x^2)), F(x), F(y), 0.5\right\} \\ &= \min\{F((x^2b)(bx^2)), F(x), F(y), 0.5\} \\ &= \min\left\{F(((bx^2)b)x^2), F(x), F(y), 0.5\right\} \\ &= \min\left\{F((((eb)x^2)b)(xx)), F(x), F(y), 0.5\right\} \\ &= \min\left\{F((((x^2b)e)b)(xx)), F(x), F(y), 0.5\right\} \\ &= \min\left\{F(((be)(x^2b))(xx)), F(x), F(y), 0.5\right\} \\ &= \min\left\{F((x^2((be)b))(xx)), F(x), F(y), 0.5\right\} \\ &\geq \min\left\{F(x^2), F(x), F(x), 0.5, F(x), F(y), 0.5\right\} \\ &\geq \min\left\{F(x), F(x), 0.5, F(x), F(x), F(x), F(y), 0.5\right\} \\ &= \min\left\{F(x), F(y), 0.5\right\}. \end{aligned}$$

Therefore, F is an $(\in, \in \vee q)$-fuzzy bi-ideal of M. □

Theorem 8. *If M is a completely regular right modular groupoid, then every $(\in, \in \vee q)$-fuzzy $(1,2)$ ideal of M is an $(\in, \in \vee q)$-fuzzy interior ideal of M.*

Proof. Let M be a completely regular and F is an $(\in, \in \vee q)$-fuzzy $(1,2)$ ideal of M.Then for $x \in M$ there exist $y \in M$ such that $x = (x^2y)x^2$, so by using $(4), (1), (2)$

and (3), we have

$$\begin{aligned} F((ax)b) &= F((a((x^2y)x^2))b) = F(((x^2y)(ax^2))b) \\ &= F((b(ax^2))(x^2y)) = F((b(a(xx)))(x^2y)) \\ &= F((b(x(ax)))(x^2y)) = F((x(b(ax)))((xx)y)) \\ &= F((x(b(ax)))((yx)x)) = F((x(yx))((b(ax))x)) \\ &= F(((ex)(yx))((x(ax))b)) = F(((xx)(ye))((a(xx))b)) \\ &= F(((xx)(ye))((b(xx))a) = F(((xx)(b(xx)))((ye)a)) \\ &= F((b((xx)(xx)))((ye)a)) = F((b(ye))(((xx)(xx))a)) \\ &= F((a((xx)(xx)))((ye)b)) = F(((xx)(a(xx)))((ye)b)) \\ &= F((((ye)b)(a(xx)))(xx)) = F((((ye)b)(x(ax)))(xx)) \\ &= F((x(((ye)b)(ax)))(xx)) = F((xc)(xx)) \\ &\geq \min\{F(x), F(x), F(x), 0.5\} = \min\{F(x), 0.5\}. \end{aligned}$$

Therefore F is an $(\in, \in \vee q)$-fuzzy interior ideal of M. □

Theorem 9. *Let F be an$(\in, \in \vee q)$ $-$fuzzy bi-ideal of a right modular groupoid M. If M is a completely regular and $F(a) < 0.5$ for all $x \in M$ then $F(a) = F(a^2)$ for all $a \in M$.*

Proof. Let $a \in M$ then there exist $x \in M$ such that $a = (a^2x)a^2$, then we have

$$\begin{aligned} F(a) &= F((a^2x)a^2) \geq \min\{F(a^2), F(a^2), 0.5\} \\ &= \min\{F(a^2), 0.5\} = F(a^2) = F(aa) \\ &\geq \min\{F(a), F(a), 0.5\} = F(a). \end{aligned}$$

Therefore $F(a) = F(a^2)$. □

Theorem 10. *Let F be an$(\in, \in \vee q)$-fuzzy interior ideal of a right modular groupoid M. If M is a completely regular and $F(a) < 0.5$ for all $x \in M$ then $F(a) = F(a^2)$ for all $a \in M$.*

Proof. Let $a \in M$ then there exist $x \in M$ such that $a = (a^2x)a^2$, using $(4), (1)$ and (3), we have

$$\begin{aligned} F(a) &= F((a^2x)a^2) = F((a^2x)(aa)) = F(a((a^2x)a)) \\ &= F(a((ax)a^2)) = F((ea)((ax)a^2)) = F((((ax)a^2)a)e) \\ &= F(((aa^2)(ax))e) = F(((xa)(a^2a))e) = F((((a^2a)a)x)e) \\ &= F((((aa)a^2)x)e) = F(((xa^2)(aa))e) = F(((xa^2)a^2)e) \\ &\geq \min\{F(a^2), 0.5\} = F(a^2) = F(aa) \\ &\geq \min\{F(a), F(a), 0.5\} \geq \min\{F(a), 0.5\} = F(a). \end{aligned}$$

Therefore $F(a) = F(a^2)$. □

$(\in, \in \vee q_k)$-fuzzy Ideals in Right Modular Groupoids

It has been given in [3] that $x_t q_k F$ is the generalizations of $x_t qF$, where k is an arbitrary element of $[0, 1)$ as $x_t q_k F$ if $F(x) + t + k > 1$. If $x_t \in F$ or $x_t qF$

implies $x_t \in q_k F$. Here we discuss the behavior of $(\in, \in \vee q_k)$-fuzzy left ideal, $(\in, \in \vee q_k)$-fuzzy right ideal,$(\in, \in \vee q_k)$-fuzzy interior ideal, $(\in, \in \vee q_k)$-fuzzy bi-ideal, $(\in, \in \vee q_k)$-fuzzy quasi-ideal in the completely regular right modular groupoid M.

Definition 10. *A fuzzy subset F of a right modular groupoid M is called an $(\in, \in \vee q_k)$-fuzzy subgroupoid of M if for all $x, y \in M$ and $t, r \in (0, 1]$ the following condition holds*

$$x_t \in F, y_r \in F \text{ implies } (xy)_{\min\{t,r\}} \in \vee q_k F.$$

Theorem 11. *Let F be a fuzzy subset of M. Then F is an $(\in, \in \vee q_k)$-fuzzy subgroupoid of M if and only if $F(xy) \geq \min\{F(x), F(y), \frac{1-k}{2}\}$.*

Proof. It is similar to the proof of theorem 1. □

Definition 11. *A fuzzy subset F of a right modular groupoid M is called an $(\in, \in \vee q_k)$-fuzzy left (right) ideal of M if for all $x, y \in M$ and $t, r \in (0, 1]$ the following condition holds*

$$y_t \in F \text{ implies } (xy)_t \in \vee q_k F \ \left(y_t \in F \text{ implies } (yx)_t \in \vee q_k F\right).$$

Theorem 12. *Let F be a fuzzy subset of M. Then F is an $(\in, \in \vee q_k)$-fuzzy left (right) ideal of M if and only if $F(xy) \geq \min\{F(y), \frac{1-k}{2}\}$ $\left(F(xy) \geq \min\{F(x), \frac{1-k}{2}\}\right)$.*

Proof. Let F be an $(\in, \in \vee q_k)$-fuzzy left ideal of M. Suppose that there exist $x, y \in M$ such that $F(xy) < \min\left\{F(y), \frac{1-k}{2}\right\}$. Choose a $t \in (0, 1]$ such that $F(xy) < t < \min\left\{F(y), \frac{1-k}{2}\right\}$. Then $y_t \in F$ but $(xy)_t \notin F$ and $F(xy) + t + k < \frac{1-k}{2} + \frac{1-k}{2} + k = 1$, so $(xy)_t \overline{\in \vee q_k} F$, a contradiction. Therefore $F(xy) \geq \min\{F(y), \frac{1-k}{2}\}$.

Conversely, assume that $F(xy) \geq \min\{F(y), \frac{1-k}{2}\}$. Let $x, y \in M$ and $t \in (0, 1]$ such that $y_t \in M$ then $F(y) \geq t$. then $F(xy) \geq \min\{F(y), \frac{1-k}{2}\} \geq \min\{t, \frac{1-k}{2}\}$. If $t > \frac{1-k}{2}$ then $F(xy) \geq \frac{1-k}{2}$. So $F(xy) + t + k > \frac{1-k}{2} + \frac{1-k}{2} + k = 1$, which implies that $(xy)_t\, q_k F$. If $t \leq \frac{1-k}{2}$, then $F(xy) \geq t$. Therefore $F(xy) \geq t$ which implies that $(xy)_t \in F$. Thus $(xy)_t \in \vee q_k F$. □

Corollary 1. *A fuzzy subset F of a right modular groupoid M is called an $(\in, \in \vee q_k)$-fuzzy ideal of M if and only if $F(xy) \geq \min\{F(y), \frac{1-k}{2}\}$ and $F(xy) \geq \min\{F(x), \frac{1-k}{2}\}$.*

Definition 12. *A fuzzy subset F of a right modular groupoid M is called an $(\in, \in \vee q_k)$-fuzzy bi-ideal of M if for all $x, y, z \in M$ and $t, r \in (0, 1]$ the following conditions hold*

(i) If $x_t \in F$ and $y_r \in M$ implies $(xy)_{\min\{t,r\}} \in \vee q_k F$,

(ii) If $x_t \in F$ and $z_r \in M$ implies $((xy)z)_{\min\{t,r\}} \in \vee q_k F$.

Theorem 13. *Let F be a fuzzy subset of M. Then F is an $(\in, \in \vee q_k)$-fuzzy bi-ideal of M if and only if*

(i) $F(xy) \geq \min\{F(x), F(y), \frac{1-k}{2}\}$ for all $x, y \in M$ and $k \in [0, 1)$,

(ii) $F((xy)z) \geq \min\{F(x), F(z), \frac{1-k}{2}\}$ for all $x, y, z \in M$ and $k \in [0, 1)$.

Proof. It is similar to the proof of theorem 1. □

Corollary 2. *Let F be a fuzzy subset of M. Then F is an $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal of M if and only if $F((xy)z) \geq \min\{F(x), F(z), \frac{1-k}{2}\}$ for all $x, y, z \in M$ and $k \in [0, 1)$.*

Definition 13. *A fuzzy subset F of a right modular groupoid M is called an $(\in, \in \vee q_k)$-fuzzy interior ideal of M if for all $x, , a, y \in M$ and $t, r \in (0,1]$ the following conditions hold*

(*i*) *If $x_t \in F$ and $y_r \in M$ implies $(xy)_{\min\{t,r\}} \in \vee q_k F$,*

(*ii*) *If $a_t \in M$ implies $((xa)\,y)_{\min\{t,r\}} \in \vee q_k F$.*

Theorem 14. *Let F be a fuzzy subset of M. Then F is an $(\in, \in \vee q_k)$-fuzzy interior ideal of M if and only if*

(*i*) *$F(xy) \geq \min\{F(x), F(y), \frac{1-k}{2}\}$ for all $x, y \in M$ and $k \in [0,1)$,*

(*ii*) *$F((xa)y) \geq \min\{F(a), \frac{1-k}{2}\}$ for all $x, a, y \in M$ and $k \in [0,1)$.*

Proof. It is similar to the proof of theorem 1. □

Lemma 3. *The intersection of any family of $(\in, \in \vee q_k)$-fuzzy interior ideals of right modular groupoid M is an $(\in, \in \vee q_k)$-fuzzy interior ideal of M.*

Proof. Let $\{F_i\}_{i \in I}$ be a family of $(\in, \in \vee q_k)$-fuzzy interior ideals of M and $x, a, y \in M$. Then $(\wedge_{i\in I} F_i)((xa)y) = \wedge_{i \in I}(F_i((xa)\,y)$. As each F_i is an $(\in, \in \vee q_k)$-fuzzy interior ideal of M, so $F_i((xa)y) \geq F_i(a) \wedge \frac{1-k}{2}$ for all $i \in I$. Thus

$$\begin{aligned}(\wedge_{i\in I} F_i)\,((xa)y) &= \wedge_{i\in I}(F_i((xa)\,y) \geq \wedge_{i\in I}\left(F_i(a) \wedge \frac{1-k}{2}\right) \\ &= (\wedge_{i\in I} F_i(a)) \wedge \frac{1-k}{2} = (\wedge_{i\in I} F_i)\,(a) \wedge \frac{1-k}{2}.\end{aligned}$$

Therefore $\wedge_{i\in I} F_i$ is an $(\in, \in \vee q_k)$-fuzzy interior ideal of M. □

Definition 14. *A fuzzy subset F of a right modular groupoid M is called an $(\in, \in \vee q_k)$-fuzzy quasi-ideal of M if following condition holds*

$$F(x) \geq \min\left\{(F \circ 1)(x), (1 \circ f)(x), \frac{1-k}{2}\right\}.$$

where 1 is the fuzzy subset of M mapping every element of M on 1.

Lemma 4. *If M is a completely regular right modular groupoid with left identity, then a fuzzy subset F is an $(\in, \in \vee q_k)$-fuzzy right ideal of M if and only if F is an $(\in, \in \vee q_k)$-fuzzy left ideal of M.*

Proof. Let F be an $(\in, \in \vee q_k)$-fuzzy right ideal of a completely regular right modular groupoid M, then for each $a \in M$ there exist $x \in M$ such that $a = (a^2x)a^2$, then by using (1), we have

$$\begin{aligned}F(ab) &= F(((a^2x)a^2)b) = F((ba^2)(a^2x)) \\ &\geq F(ba^2) \wedge \frac{1-k}{2} \geq F(b) \wedge \frac{1-k}{2}.\end{aligned}$$

Conversely, assume that F is an $(\in, \in \vee q_k)$-fuzzy right ideal of M, then by using (1), we have

$$\begin{aligned}F(ab) &= F(((a^2x)a^2)b) = F((ba^2)(a^2x)) \\ &\geq F(a^2x) \wedge \frac{1-k}{2} = F((aa)x) \wedge \frac{1-k}{2} \\ &= F((xa)a) \wedge \frac{1-k}{2} \geq F(a) \wedge \frac{1-k}{2}.\end{aligned}$$

□

Theorem 15. *If M is a completely regular right modular groupoid with left identity, then a fuzzy subset F is an $(\in, \in \vee q_k)$-fuzzy ideal of M if and only if F is an $(\in, \in \vee q_k)$-fuzzy interior ideal of M.*

Proof. Let F be an $(\in, \in \vee q_k)$-fuzzy interior ideal of a completely regular right modular groupoid M, then for each $a \in M$ there exist $x \in M$ such that $a = (a^2 x)a^2$, then by using (4) and (1), we have

$$F(ab) = F(((a^2x)a^2)b) \geq F(aa) \geq F(a) \wedge F(a) \wedge \frac{1-k}{2}, \text{ and}$$

$$\begin{aligned} F(ab) &= F(a((b^2y)b^2))F((b^2y)(ab^2)) = F(((bb)y)(ab^2)) \\ &= F(((yb)b)(ab^2)) \geq F(b) \wedge \frac{1-k}{2}. \end{aligned}$$

The converse is obvious. □

Theorem 16. *If M is a completely regular right modular groupoid with left identity, then a fuzzy subset F is an $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal of M if and only if F is an $(\in, \in \vee q_k)$-fuzzy bi-ideal of M.*

Proof. Let F be an $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal of a completely regular right modular groupoid M, then for each $a \in M$ there exist $x \in M$ such that $a = (a^2x)a^2$, then by using (4), we have

$$\begin{aligned} F(ab) &= F(((a^2x)a^2)b) = F(((a^2x)(aa))b) \\ &= F((a((a^2x)a))b) \geq F(a) \wedge F(b) \wedge \frac{1-k}{2}. \end{aligned}$$

The converse is obvious. □

Theorem 17. *If M is a completely regular right modular groupoid with left identity, then a fuzzy subset F is an $(\in, \in \vee q_k)$-fuzzy bi-ideal of M if and only if F is an $(\in, \in \vee q_k)$-fuzzy two sided ideal of M.*

Proof. Let F be an $(\in, \in \vee q_k)$-fuzzy bi-ideal of a completely regular right modular groupoid M, then for each $a \in M$ there exist $x \in M$ such that $a = (a^2x)a^2$, then by using (1) and (4), we have

$$\begin{aligned} F(ab) &= F(((a^2x)a^2)b) = F((((aa)x)a^2)b) = F((((xa)a)a^2)b) \\ &= F((ba^2)((xa)a)) = F((b(aa))((aa)x)) = F((a(ba))((aa)x)) \\ &= F((aa((a(ba))x)) = F((((a(ba))x)a)a) = F((((b(aa))x)a)a) \\ &= F((((x(aa))b)a)a) = F((((a(xa))b)a)a) = F(((ab)(a(xa)))a) \\ &= F((a((ab)(xa)))a) \geq F(a) \wedge F(a) \wedge \frac{1-k}{2} = F(a) \wedge \frac{1-k}{2}. \end{aligned}$$

And, by using (4) , (1) and (3),we have

$$\begin{aligned} F(ab) &= F(a((b^2y)b^2)) = F((b^2y)(ab^2)) = F(((bb)y)(a(bb))) \\ &= F(((a(bb)y)(bb)) = F(((a(bb))(ey))(bb)) = F(((ye)((bb)a))(bb)) \\ &= F((bb)((ye)a))(bb)) \geq F(bb) \wedge \frac{1-k}{2}. \geq F(b) \wedge F(b) \wedge \frac{1-k}{2}. \\ &= F(b) \wedge \frac{1-k}{2}. \end{aligned}$$

The converse is obvious. □

Theorem 18. *If M is a completely regular right modular groupoid with left identity, then a fuzzy subset F is an $(\in, \in \vee q_k)$-fuzzy quasi-ideal of M if and only if F is an $(\in, \in \vee q_k)$-fuzzy two sided ideal of M.*

Proof. Let F be an $(\in, \in \vee q_k)$-fuzzy quasi-ideal of a completely regular right modular groupoid M, then for each $a \in M$ there exist $x \in M$ such that $a = (a^2x)a^2$, then by using (1), (3) and (4), we have

$$\begin{aligned} ab &= ((a^2x)a^2)b = (ba^2)\left(a^2x\right) = \left(xa^2\right)\left(a^2b\right) \\ &= (x\,(aa))\left(a^2b\right) = (a\,(xa))\left(a^2b\right) = \left(\left(a^2b\right)(xa)\right)a. \end{aligned}$$

Then

$$\begin{aligned} F(ab) &\geq (F \circ 1)(ab) \wedge (1 \circ F)(ab) \wedge \frac{1-k}{2} \\ &= \bigvee_{ab=pq} \{F(p) \wedge 1\,(q)\} \wedge (1 \circ F)(ab) \wedge \frac{1-k}{2} \\ &\geq F\,(a) \wedge 1\,(b) \wedge \bigvee_{ab=lm} \{F\,(l) \wedge 1(m)\} \wedge \frac{1-k}{2} \\ &= F\,(a) \wedge \bigvee_{ab=((a^2b)(xa))a} \left\{1\left(\left(a^2b\right)(xa)\right) \wedge F\,(a)\right\} \\ &\geq F\,(a) \wedge 1\left(\left(a^2b\right)(xa)\right) \wedge F\,(a) \wedge \frac{1-k}{2} = F\,(a) \wedge \frac{1-k}{2}. \end{aligned}$$

Also by using (4) and (1), we have

$$\begin{aligned} ab &= a\left(\left(b^2y\right)b^2\right) = \left(b^2y\right)\left(ab^2\right) = ((bb)\,y)\left(ab^2\right) \\ &= \left(\left(ab^2\right)y\right)(bb) = b\left(\left(\left(ab^2\right)y\right)b\right). \end{aligned}$$

Then

$$\begin{aligned} F(ab) &\geq (F \circ 1)(ab) \wedge (1 \circ F)(ab) \wedge \frac{1-k}{2} \\ &= \bigvee_{ab=pq} \{F(p) \wedge 1\,(q)\} \wedge \bigvee_{ab=lm} \{1(l) \wedge F\,(m)\} \wedge \frac{1-k}{2} \\ &= \bigvee_{ab=b(((ab^2)y)b)} \left\{F(b) \wedge 1(\left(\left(ab^2\right)y\right)b)\right\} \wedge \bigvee_{ab=ab} \{1(a) \wedge F(b)\} \wedge \frac{1-k}{2} \\ &\geq F\,(b) \wedge 1(\left(\left(ab^2\right)y\right)b) \wedge 1\,(a) \wedge F\,(b) \wedge \frac{1-k}{2} = F\,(b) \wedge \frac{1-k}{2}. \end{aligned}$$

The converse is obvious □

Remark 1. *We note that in a completely regular right modular groupoid M with left identity, $(\in, \in \vee q_k)$-fuzzy left ideal $(\in, \in \vee q_k)$-fuzzy right ideal, $(\in, \in \vee q_k)$ fuzzy ideal,$(\in, \in \vee q_k)$-fuzzy interior ideal, $(\in, \in \vee q_k)$-fuzzy bi-ideal, $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal and $(\in, \in \vee q_k)$-fuzzy quasi-ideal coincide with each other.*

Theorem 19. *If M is a completely regular right modular groupoid then $F \wedge_k G = F \circ_k G$ for every $(\in, \in \vee q_k)$- fuzzy ideal F and G of M.*

Proof. Let F is an $(\in, \in \vee q_k)$- fuzzy right ideal of M and G is an $(\in, \in \vee q_k)$-fuzzy left ideal of M, and M is a completely regular then for each $a \in M$ there exist $x \in M$ such that $a = (a^2x)a^2$, so we have

$$\begin{aligned}
(F \circ_k G)(a) &= (F \circ G)(a) \wedge \frac{1-k}{2} = \bigvee_{a=pq} \{F(p) \wedge G(q)\} \wedge \frac{1-k}{2} \\
&\geq F(a^2x) \wedge G\left(a^2\right) \wedge \frac{1-k}{2} \geq F(aa) \wedge G(aa) \wedge \frac{1-k}{2} \\
&\geq F(aa) \wedge G(aa) \wedge \frac{1-k}{2} \geq F(a) \wedge G(a) \wedge \frac{1-k}{2} \\
&= (F \wedge G)(a) \wedge \frac{1-k}{2} = (F \wedge_k G)(a).
\end{aligned}$$

Therefore $F \wedge_k G \leq F \circ_k G$, again

$$\begin{aligned}
(F \circ_k G)(a) &= (F \circ G)(a) \wedge \frac{1-k}{2} = \left(\bigvee_{a=pq} \{F(p) \wedge G(q)\}\right) \wedge \frac{1-k}{2} \\
&= \bigvee_{a=pq} \left\{F(p) \wedge G(q) \wedge \frac{1-k}{2}\right\} \leq \bigvee_{a=pq} \left\{(F(pq \wedge G(pq)) \wedge \frac{1-k}{2}\right\} \\
&= F(a) \wedge G(a) \wedge \frac{1-k}{2} = (F \wedge_k G)(a).
\end{aligned}$$

Therefore $F \wedge_k G \geq F \circ_k G$. Thus $F \wedge_k G = F \circ_k G$. $\square$

Definition 15. *A right modular groupoid M is called weakly regular if for each a in M there exists x and y in M such that $a = (ax)(ay)$.*

It is easy to see that right regular, left regular and weakly regular coincide in a right modular groupoid with left identity.

Theorem 20. *For a weakly regular right modular groupoid M with left identity, $(G \wedge_k F) \wedge_k H) \leq ((G \circ_k F) \circ_k H)$, where G is an $(\in, \in \vee q_k)$- fuzzy right ideal, F is an $(\in, \in \vee q_k)$- fuzzy interior ideal and H is an $(\in, \in \vee q_k)$- fuzzy left ideal.*

Proof. Let M be a weakly regular right modular groupoid with left identity, then for each a in M there exists x and y in M such that $a = (ax)(ay)$, then by using (3), we get $a = (ya)(xa)$, and also by using (4) and (3), we get

$$ya = y((ax)(ay)) = (ax)(y(ay)) = (ax)((ey)(ay)) = (ax)((ya)(ye))$$

Then

$$\begin{aligned}((G \circ_k F) \circ_k H)(a) &= \bigvee_{a=pq} \{(G \circ_k F)(p) \wedge H(q)\} \geq (G \circ_k F)(ya) \wedge H(xa) \\ &\geq (G \circ_k F)(ya) \wedge H(a) \wedge \frac{1-k}{2} \\ &= \bigvee_{ya=bc} \{G(b) \wedge F(c)\} \wedge H(a) \wedge \frac{1-k}{2} \\ &\geq (G(ax) \wedge F((ya)(ye))) \wedge H(a) \wedge \frac{1-k}{2} \\ &\geq (G(a) \wedge \frac{1-k}{2} \wedge F(a) \wedge \frac{1-k}{2}) \wedge H(a) \wedge \frac{1-k}{2} \\ &= (G(a) \wedge F(a)) \wedge H(a) \wedge \frac{1-k}{2} \\ &= ((G \wedge F) \wedge H)(a) \wedge \frac{1-k}{2} \\ &= ((G \wedge_k F) \wedge_k H)(a).\end{aligned}$$

Therefore, $(G \wedge_k F) \wedge_k H) \leq ((G \circ_k F) \circ_k H)$. □

Theorem 21. *For a weakly regular right modular groupoid M with left identity, $F_k \leq ((F \circ_k 1) \circ_k F)$, where F is an $(\in, \in \vee q_k)$- fuzzy interior ideal.*

Proof. Let M be a weakly regular right modular groupoid with left identity, then for each $a \in M$ there exist $x, y \in M$ such that $a = (ax)(ay)$, then by using (1) $a = ((ay)x)a$. Also by using (1) and (4), we have

$$\begin{aligned}(ay) &= (((ax)(ay))y) = ((y(ay))(ax)) = ((a(yy))(ax)) \\ &= (((ax)(yy))a) = ((((yy)x)a)a).\end{aligned}$$

Then

$$\begin{aligned}((F \circ_k 1) \circ_k F)(a) &= ((F \circ 1) \circ F)(a) \wedge \frac{1-k}{2} \\ &= \bigvee_{a=pq} \{(F \circ 1)(p) \wedge F(q)\} \wedge \frac{1-k}{2} \\ &\geq (F \circ 1)(((ay)x) \wedge F(a) \wedge \frac{1-k}{2} \\ &= \bigvee_{(ay)x=(bc)} \{F(b) \wedge 1(c)\} \wedge F(a) \wedge \frac{1-k}{2} \\ &\geq F(ay) \wedge 1(x) \wedge F(a) \wedge \frac{1-k}{2} \\ &= F((((yy)x)a)a) \wedge F(a) \wedge \frac{1-k}{2} \\ &\geq F(a) \wedge \frac{1-k}{2} \wedge F(a) \wedge \frac{1-k}{2} \\ &= F(a) \wedge \frac{1-k}{2} = F_k(a).\end{aligned}$$

Therefore, $F_k \leq ((F \circ_k 1) \circ_k F)$. □

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